Perturbation Analysis of Bistability and Period Doubling Bifurcations in Directly-Modulated Laser Diodes

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Abstract—Resonance frequency shift, bistability, and period doubling bifurcations in directly-modulated semiconductor lasers are accounted for by applying multiple scale expansions to the rate equations that take into account spontaneous emission and gain saturation. Explicit algebraic expressions of parameter ranges for resonance frequency shift, bistability, and period 2 and period 4 oscillations are derived, and they are compared with results of numerical calculations. We also explain physical mechanisms of resonance frequency shift and period doubling bifurcations.

I. INTRODUCTION

CHAOTIC behavior in nonautonomous laser systems has been studied theoretically and experimentally by a number of authors [1]–[9]. Recently, chaos in pump-modulated lasers has been of considerable interest [3]–[9]. In demonstrating the existence of chaotic behavior in pump-modulated lasers, semiconductor lasers will provide a good example because they can be modulated directly by the injection current. It was theoretically shown by the authors [4] that directly-modulated semiconductor lasers exhibit period doublings and chaos. Later this was demonstrated by experiment [5]. The effects of spontaneous emission [4], noise [5], [6], gain saturation [7], and Auger recombination [8] on the chaotic behavior in the semiconductor lasers were also studied. Chaos in pump-modulated lasers was also observed experimentally in a solid-state laser [3] and a Nd-doped fiber laser [9].

In this paper the resonance frequency shift, bistability, and period doubling bifurcations in a directly-modulated laser diode are accounted for by solving analytically the full rate equations that take into account the gain saturation and the spontaneous emission [10]. In Section II the rate equations are solved systematically by the method of multiple scales, when the frequency of the drive current is close to the small-signal resonance frequency of the rate equations. Thus it is shown that the resonance peak in the frequency response shifts toward lower frequencies as the modulation current is increased, and that it may accompany a hysteresis phenomenon. Explicit algebraic expressions for peak amplitude, resonance frequency shift, and bistability under large-signal modulation are derived. Similar results based on a perturbation method were derived previously by Harth [11] in a different way from ours. His result was obtained by neglecting both the spontaneous emission factor and the gain saturation coefficient which affect seriously the dynamic behavior of a laser diode. It was also expressed by a transcendental equation that is solvable by a numerical method. In Section III the first subharmonic solutions of rate equations are obtained systematically by the method of multiple scales. Parameter ranges for generation of the first subharmonic solution are given. Ranges where two attractors (period 1 and period 2) coexist are also given. Harth and Siemons [12] have shown analytically the first subharmonic generation neglecting both the spontaneous emission and the gain saturation in the rate equations. However, it was not realized that such a subharmonic solution forms the onset of the period doubling route to chaos [4]. They also did not give an explicit expression for subharmonic amplitude. In Section IV, a mechanism for the generation of the higher subharmonic is accounted for by considering a period 4 solution. The condition for a period 4 solution is also obtained. Finally, discussions and the conclusions are given in Section V.

II. RESONANCE FREQUENCY SHIFT AND BISTABILITY

The dynamics of laser diodes may be described in terms of the following normalized nonlinear rate equations [7]:

\[ \frac{dn}{d\tau} = i - (1 - \epsilon_g s)(n - n_s) - n, \]

\[ \frac{ds}{d\tau} = \tau[(1 - \epsilon_g s)(n - n_s) + \beta n] \quad (1a) \]
where the electron density \( n = g_e \tau_e N \), the photon density \( s = g_p \tau_p S \), and the injection current \( i = (g_e \tau_e \tau_p/V)/e I \). \( N \), \( S \), and \( I \) represent the respective non-normalized quantities. The unsaturated gain coefficient \( g_e \), the spontaneous electron lifetime \( \tau_e \), the photon lifetime \( \tau_p \), the volume of active region \( V \), the normalized gain saturation coefficient \( \epsilon_e \), and the spontaneous emission factor \( \beta \) are laser diode parameters. \( n_e \) is the minimum electron density required to obtain positive gain and \( e \) is the electron charge. Time is also normalized as \( \tau = t/\tau_e \), and \( T = \tau_p/\tau_e \). The injection current \( i \) may be represented by the bias current \( i_b \) plus the modulation current \( i_m \).

The normalized rate equations may be transformed to the generalized coordinates \( [10, 13] \). \( x = n_1 \) and \( y = (\omega_0 x_1 + s_1)/\omega_0 \), where \( n_1 \) and \( s_1 \) are increments of photon density and electron density from steady-state solutions of rate equations with a constant injection current \( i_b \):

\[
\frac{dx}{d\tau} = i_m - \alpha x - \omega_0 y - \omega_0 xy, \quad (2a)
\]

\[
\frac{dy}{d\tau} = -\alpha y + \omega_0 x + Txy + \frac{\alpha}{\omega_0} i_m. \quad (2b)
\]

Here, \( \alpha = 1 + s_0 - T(\epsilon_e s_0 + \beta(1 + n_0)/s_0) \). \( 2\alpha = 1 + s_0 + T(\epsilon_e s_0 + \beta(1 + n_0)/s_0) \). \( \omega_0 = (Ts_0)^{1/2} \), and \( s_0 = i_b - 1 - n_0 \) is the steady-state photon density. The steady-state electron density \( n_0 \) does not appear explicitly in (2), since we used an expression \( n_0 = 1 + n_0 + \epsilon_e s_0 - \beta(1 + n_0)/s_0 \). Details of this transformation and its advantages are discussed in [13]. \( \omega_0 \) and \( \alpha \) are the angular resonance frequency and the damping constant of the linearized rate equations, respectively. Both the spontaneous emission and the gain saturation increase the damping constant [14]. For semiconductor lasers, typical values of parameters are \( T \approx 1000, \omega_0 \approx 1, \epsilon_e \approx 10^{-3} \), and \( \beta \approx 10^{-4} \). Thus \( \alpha \approx 1 \) and \( \omega_0 \approx 30 \). In (2) we neglected small nonlinear terms involved with \( \epsilon_e \) or \( \alpha/\omega_0 \), since \( |\alpha|/\omega_0 < \alpha/\omega_0 \ll 1 \). However, we retained \( \alpha i_m/\omega_0 \) term since \( i_m \) is a driving force term.

In (2), only two parameters \( \alpha \) and \( \alpha/\omega_0 \) depend on \( \beta \) and \( \epsilon \). Thus the effects of spontaneous emission and gain saturation on the large signal response of the laser diode are qualitatively equivalent in the transformed nonlinear rate equations. (2). It is important to note that the rate equations with the Auger recombination term can also be transformed to (2), and the Auger recombination term increases the damping constant slightly [13]. In other words, the effects of the spontaneous emission, the gain saturation, and the Auger recombination are qualitatively equivalent in the transformed rate equations.

To solve (2) by using the perturbation theory, we introduce dynamic variables, \( u = Tx \), \( v = Ty \), \( \xi = \omega_0 \tau \), and a parameter \( \epsilon = 1/\omega_0 \approx 1/30 \). Regarding \( \epsilon \) as a small parameter and keeping up to the first-order terms of \( \epsilon \) for \( u \) and \( v \), we obtain the following second-order nonlinear equation [13]

\[
\frac{du}{d\xi} + 2\epsilon \alpha \frac{du}{d\xi} + \left( 1 + \frac{i_m}{i_b} \right) u - \epsilon \frac{du}{d\xi} = \frac{T}{\omega_0} \frac{di_m}{d\xi} + \frac{i_m}{s_0} (\alpha - \hat{\alpha}). \quad (3)
\]

In (3) we retained all the source terms involving \( i_m \), since their order estimation will depend on problems of our interest. The modulation current is given by \( i_m = i_m \rho m \cos \omega' \xi \), where \( \rho m \) is the modulation index and \( \omega' \) is the normalized angular modulation frequency (i.e., the angular modulation frequency \( \omega \) divided by \( \omega_0 \)). Here, we notice that \( u \) represents the scaled normalized electron density. Equation (3) describes excitation by both external and parametric forces and nonlinearity is introduced by \( u du/d\xi \). It has been found that nonlinear oscillators excited by an external force [15], [16] or parametrically [17] may exhibit a period doubling route to chaos.

Now we apply the method of multiple scales [13], [18], a standard perturbation technique, to (3) to study resonance frequency shift and possible bistability in a directly-modulated laser diode. For that, we investigate the modulation characteristics of the laser diode at the frequency close to the small-signal resonance frequency, i.e., \( \omega' = 1 + \epsilon \sigma \), where \( \sigma = O(1) \) is a detuning factor. When \( \sigma = 0 \), neglection of the damping in the linearized version of (3) will predict unbounded oscillations. However, the actual oscillator attains a finite amplitude due to the damping and the nonlinearity. To obtain a uniformly-valid approximate solution, we have to order the largest excitation term so that in perturbation expansions it will appear when the damping and the nonlinearity appear. For that purpose, it may be regarded that the largest driving term \( T \omega_0^{-1} di_m/d\xi \) in (3) is of the order of \( \epsilon \) (see [18, 4.1]). Thus we may rewrite (3) for \( m = O(\epsilon^2) \) as

\[
\frac{du}{d\xi} + 2\epsilon \alpha \frac{du}{d\xi} + \left( 1 + \epsilon^2 c_1 \cos \omega' \xi \right) u - \epsilon \frac{du}{d\xi} = \epsilon c_1 \frac{d}{d\xi} \cos \omega' \xi + \epsilon^2 c_1 (\alpha - \hat{\alpha}) \cos \omega' \xi \quad (4)
\]

where \( c_1 = T\omega_0 m (= O(1)) \). With \( \epsilon \) as a small parameter we seek an expansion up to the second order as

\[
u(\xi; \epsilon) = u_0(\xi_0, \xi_1, \xi_2) + \epsilon u_1(\xi_0, \xi_1, \xi_2)
\]

\[+ \epsilon^2 u_2(\xi_0, \xi_1, \xi_2) \quad (5a)
\]

\[
\frac{d}{d\xi} = \frac{\partial}{\partial \xi_0} + \epsilon \frac{\partial}{\partial \xi_1} + \epsilon^2 \frac{\partial}{\partial \xi_2} \quad (5b)
\]

where \( \xi_n = \epsilon^n \xi \). Substituting (5) into (4) and equating
coefficients of like powers of \( \epsilon \), we obtain
\[
\frac{\partial^2 u_0}{\partial \xi_0^2} + u_0 = 0
\]
\[
\frac{\partial^2 u_1}{\partial \xi_1^2} + u_1 = -2 \frac{\partial u_0}{\partial \xi_0} \frac{\partial u_1}{\partial \xi_1} - 2\alpha \frac{\partial^2 u_0}{\partial \xi_0^2} + \frac{1}{2} \frac{\partial^2 u_0}{\partial \xi_0^2} + c_1 \sin (\omega' \xi)
\]
\[
\frac{\partial^2 u_2}{\partial \xi_0^2} + u_2 = -2 \frac{\partial u_0}{\partial \xi_0} \frac{\partial u_1}{\partial \xi_2} - 2 \frac{\partial^2 u_1}{\partial \xi_0^2} - \frac{\partial^2 u_0}{\partial \xi_1^2}
\]
\[
-2\alpha \left( \frac{\partial u_1}{\partial \xi_0} + \frac{\partial u_0}{\partial \xi_1} \right)
\]
\[
+ \frac{1}{2} \frac{\partial^2 u_1}{\partial \xi_1^2} + c_1 \frac{\partial^2 u_0}{\partial \xi_0^2} - \sigma c_1 \sin (\omega' \xi)
\]
\[
+ c_1 (\alpha - \dot{\alpha}) \cos (\omega' \xi) - c_1 \cos (\omega' \xi) u_0
\]
\[
(6c)
\]

The general solution of (6a) may be written as
\[
u_0 = F(\xi_1, \xi_2) e^{i\phi} + c.c.
\]
where c.c. stands for the complex conjugate of the preceding term. Substituting (7) into (6b) and eliminating the terms proportional to \( e^{i\phi} \) that produce secular terms in the particular solution of \( u_1 \), we obtain
\[
\frac{\partial F}{\partial \xi_2} + \alpha F - \frac{c_1}{4} e^{i\phi} = 0.
\]
\[
(8)
\]
Then, the particular solution of (6b) is given by
\[
u_1 = -\frac{j}{3} F^2 e^{i2\xi_2} + c.c.
\]
\[
(9)
\]
Substituting \( u_0 \) and \( u_1 \) into (6c) and eliminating similarly the terms that produce secular terms, we obtain
\[
-2\frac{j}{3} \frac{\partial F}{\partial \xi_2} - \frac{\partial F}{\partial \xi_1} - 2\alpha \frac{\partial F}{\partial \xi_1} + \frac{F^2 F^*}{3}
\]
\[
+ \frac{c_1}{2} \left( j \sigma + \alpha - \dot{\alpha} \right) e^{i\phi} = 0.
\]
\[
(10)
\]

It can be shown by following a standard example (see [18, p. 293]) that (8) and (10) result from a multiple-scale expansion of
\[
-2\frac{j}{3} \frac{\partial F}{\partial \xi} + j \left( \frac{c_1}{2} e^{i\omega t} - 2\alpha F \right) + \epsilon^2 \left[ \alpha^2 F + \frac{1}{3} F^2 F^* \right]
\]
\[
+ c_1 \left( \frac{\sigma}{4} + \alpha - \frac{\dot{\alpha}}{2} \right) e^{i\phi} = 0.
\]
\[
(11)
\]

From now on we will neglect the \( \alpha^2 F \) term in (11), since this term remains uncanceled due to an approximation of the resonance angular frequency \( 1 + \epsilon^2 \omega' \) as unity in deriving (3) to the first order of \( \epsilon \). We seek a solution of (11) as
\[
F = \frac{1}{2} a(\xi) e^{i\sigma t + \phi(\xi)}.
\]
\[
(12)
\]

Substituting (12) into (11) and separating real and imaginary parts, we have
\[
\frac{da}{d\xi} = -a \sigma + c_1 \left( 1 + \frac{\sigma}{2} \right) \sin \phi
\]
\[
- \frac{\epsilon^2 c_1}{4} (\alpha - 2 \dot{\alpha}) \sin \phi
\]
\[
(13a)
\]
\[
-a \frac{d\phi}{d\xi} = a \sigma + \epsilon^2 a \frac{a^2}{24} + \frac{c_1}{2} \left( 1 + \frac{\sigma}{2} \right) \sin \phi
\]
\[
+ \frac{\epsilon^2 c_1}{4} (\alpha - 2 \dot{\alpha}) \cos \phi.
\]
\[
(13b)
\]

To determine the steady-state solution, we set \( da/d\xi = 0 \) and \( d\phi/d\xi = 0 \). Then, we obtain by eliminating \( \phi \) in (13)
\[
a^2 \left[ \left( \sigma + \frac{\epsilon^2}{24} a^2 \right)^2 + \alpha^2 \right] = \frac{T_{\text{im}} m^2}{4} \left( 1 + \frac{\sigma}{2} \right).
\]
\[
(14)
\]

Equation (14) is an expression for the amplitude \( a \) as a function of the driving frequency, i.e., the frequency response.

From (14), the peak amplitude \( a_p \) in the frequency response may be given approximately as
\[
a_p^2 = \frac{a_{p, \text{pl}}^2}{1 + \epsilon^2 a_{p, \text{pl}}^2/24} \approx a_{p, \text{pl}}^2, \quad a_{p, \text{pl}} = \frac{T_{\text{im}} m}{2 \alpha}
\]
\[
(15)
\]

where \( a_{p, \text{pl}} \) is the peak amplitude given by the linearized equation of (4). It decreases with an increase of the damping constant \( \alpha \) as well known from linear analysis. As the modulation index \( m \) is increased, \( a_p \) becomes smaller than \( a_{p, \text{pl}} \). The resonance frequency shift \( \Delta \omega' \) due to modulation can be written as
\[
\Delta \omega' = -\epsilon^2 a_p^2
\]
\[
\frac{24}{24} \approx \frac{(T_{\text{im}} m)^2}{96e^2 \alpha^2}
\]
\[
(16)
\]

by deriving \( \sigma \) at \( a = a_p \) from (14). Magnitude of the resonance frequency shift increases in proportion to the square of the modulation current amplitude (i.e., the modulation power) and it is inversely proportional to the square of the damping constant \( \alpha \). The resonance frequency shift increases with an increase of \( T \).

This shift of resonance frequency is the same as the resonance frequency shift in the transient solution [13]. With an increase of the modulation current, the peak of the photon density increases. In other words, the carrier density depletes far below the threshold. Then, required time to recover the carrier density above the threshold value becomes longer. Thus the larger peak photon density requires a longer modulation period, i.e., the resonance frequency shifts toward the lower frequency side with increase of the modulation power.

Frequency response curves obtained from (14) are plotted for various values of \( m \) and \( \alpha \) in Fig. 1. As the modulation index \( m \) is increased, the resonance peak shifts
which is closely dependent on the spontaneous emission factor $\beta$ and the gain saturation coefficient $\epsilon_{g}$, is crucial to the dynamic behavior of laser diodes.

Finally, it may be noted that resonance frequency shift has been observed experimentally [5]. Hysteresis has not been observed yet in a laser diode, but in a Nd-doped fiber laser [9]. Typical parameters for the Nd-doped fiber laser are $T \sim 5 \times 10^{4}$, $n_{u} = 0$, and $\epsilon_{g} = \beta = 0$. Negligible gain saturation and spontaneous emission explain relatively easy observation of hysteresis (or bistability).

III. FIRST SUBHARMONIC GENERATION AND BISTABILITY

It is well known that generation of a period 2 solution or the first period doubling bifurcation in a directly-modulated laser diode is possible by subharmonic resonance occurring when the frequency of the driving current is close to two times the small-signal resonance frequency of the rate equations [12]. To understand these features quantitatively, we perform multiple scales expansion of (3) with $\omega' = 2 + \epsilon\sigma$, where $\sigma$ is a detuning factor. Since $\omega'$ is away from 1 (i.e., small-signal resonance frequency), the effect of the excitation will be small unless the largest driving term is of the order of 1. Thus the largest driving term $T_{\omega}^{-1} d_{u}/d_{\xi}$ in (3) may be regarded to be of the order of 1 (see [18, 4.1]). Then, it is required that $m = O(\epsilon)$. Rewriting (3)

$$\frac{d^{2}u}{d\xi^{2}} + 2\epsilon\sigma \frac{du}{d\xi} + \left[1 + \epsilon c_{2} \cos(\omega'\xi)\right]u - \epsilon u \frac{du}{d\xi}$$

$$= c_{2} \frac{d}{d\xi} \cos(\omega'\xi) + \epsilon c_{2}(\alpha - \tilde{\alpha}) \cos(\omega'\xi)$$

(18) where $c_{2} = T_{\omega} m / \omega_{\alpha}(= O(1))$. There exist two parametric excitations. One is the direct excitation of $\epsilon c_{2} \cos(\omega'\xi)$ and the other is the indirect excitation due to the nonlinear term $\epsilon u \frac{du}{d\xi}$. They compete with each other, but the attainment of a finite amplitude of the parametric oscillation involves nonlinear effects due to the latter term. Substituting (5) into (18) and equating coefficients of like powers of $\epsilon$, we obtain a set of equations.

$$\frac{\partial^{2}u_{0}}{\partial\xi_{0}^{2}} + u_{0} = -2c_{2} \sin(\omega'\xi)$$

(19a)

$$\frac{\partial^{2}u_{1}}{\partial\xi_{0}^{2}} + u_{1} = -2 \left(\frac{\partial^{2}u_{0}}{\partial\xi_{0}^{2}} - 2\alpha \frac{\partial u_{0}}{\partial\xi_{0}} + \frac{1}{2} \frac{\partial^{2}u_{0}}{\partial\xi_{0}^{2}}\right)$$

$$- c_{2}\sigma \sin(\omega'\xi) + c_{2}(\alpha - \tilde{\alpha} - \gamma_{u}) \cos(\omega'\xi)$$

(19b)

$$\frac{\partial^{2}u_{2}}{\partial\xi_{0}^{2}} + u_{2} = -2 \frac{\partial^{2}u_{0}}{\partial\xi_{0}^{2}} - 2 \left(\frac{\partial^{2}u_{1}}{\partial\xi_{0}^{2}} - \frac{\partial^{2}u_{0}}{\partial\xi_{0}^{2}}\right) - \frac{\partial^{2}u_{1}}{\partial\xi_{1}^{2}}$$

$$- 2\alpha \left(\frac{\partial u_{1}}{\partial\xi_{0}} + \frac{\partial w_{0}}{\partial\xi_{0}}\right)$$

$$+ \frac{1}{2} \frac{\partial^{2}u_{1}}{\partial\xi_{1}^{2}} - c_{2} \cos(\omega'\xi)u_{1}.$$  

(19c)
The general solution of (19a) may be written as
\[ u_0 = G(\xi_1, \xi_2) e^{i\Omega t} - j \frac{c_5}{2} e^{i\omega t} + \text{c.c.} \] (20)
where the complex amplitude \( G \) of the homogeneous solution of \( u_0 \) is the amplitude of subharmonic solution. Following the same procedure as in Section II, we eliminate secular terms in particular solutions (19b) and (19c) to derive
\[ -2j \frac{\partial G}{\partial \xi} - \epsilon \left[ 2\alpha G + \frac{c_2}{6} G^* e^{i\omega t} \right] + \epsilon^2 \left[ \alpha^2 G + \frac{5c_2^2}{288} G + \frac{G^2G^*}{3} + \frac{c_5G^*}{3} e^{i\omega t} \right] = 0. \] (21)
As we did for (11), we will neglect the \( \alpha^2 G \) term in (21) from now on. We seek a solution of (21) such that
\[ G = b_0(\xi) e^{i\Omega t/2 + \theta(\xi)}. \] (22)
Substituting (22) into (21) and separating real and imaginary parts, we obtain
\[ \frac{db}{d\xi} = -\alpha eb + \epsilon c_2b \left( \frac{1}{12} - \frac{5\epsilon}{72} \right) \sin 2\theta \]
\[ + \epsilon^2 c_2b \frac{22\alpha + 6\epsilon}{72} \cos 2\theta \] (23a)
\[ -b \frac{d\theta}{d\xi} = \epsilon bX - \epsilon c_2b \left( \frac{1}{12} - \frac{5\epsilon}{72} \right) \cos 2\theta \]
\[ + \epsilon^2 c_2b \frac{22\alpha + 6\epsilon}{72} \sin 2\theta \] (23b)
where \( X = \frac{\sigma}{2} + \epsilon \left\{ 5c_2^2/576 + \frac{e^2}{24} \right\} \). The steady-state solution (23) yields
\[ b^2 \left[ \left( \frac{\sigma}{2} + \epsilon \left( \frac{5c_2^2}{576} + \frac{\epsilon^2}{24} \right) \right) + \alpha^2 \right] \]
\[ \approx b^2 \left[ \frac{c_2^2}{12} \left( 1 - \frac{5\epsilon}{3} \right) \right]. \] (24)
The terms \( \epsilon b^2/24 \) and \( \epsilon \left( 5/576 \right) c_2^2 \) in (24) represent a subharmonic-amplitude-dependent resonance frequency shift and a driving-force-dependent resonance frequency shift, respectively. Solutions of (24) are given by
\[ b^2 = 0 \] (25a)
\[ b^2 = -12 \left[ \frac{\sigma}{\epsilon} + \frac{5c_2^2}{288} \right] \pm 2 \epsilon \sqrt{c_2^2 \left( 1 - \frac{5\epsilon}{3} \right) - \left( 12\alpha \right)^2}. \] (25b)
We consider two types of solutions for \( b^2 > 0 \) in (25), depending on whether \( b^2 \) is always single-valued or not, as shown in Fig. 2. For small values of \( m \), only the solution \( b = 0 \) is possible, i.e., there is no subharmonic resonance. In case of \( \omega' > 2 \), the subharmonic solution exists for \( m > m_1 \) and \( b^2 \) is always single-valued [see Fig. 2 (a)]. Here \( m_1 \) is obtained from the condition for \( b^2 \) to be nonnegative (i.e., \( b = 0 \) at \( m = m_1 \)). On the other hand, in case of \( \omega' < 2 \), the subharmonic solution exhibits hysteresis for \( m_2 < m < m_1 \), as shown in Fig. 2 (b). As we increase the modulation index from zero, there is no subharmonic resonance (i.e., \( b = 0 \)) until \( m = m_2 \). At that point the subharmonic amplitude \( b \) suddenly takes a finite value jumping to the upper branch. As we decrease the modulation index on the upper branch, the amplitude of the subharmonics decreases and it falls to zero at \( m = m_2 \) abruptly. Thus the existence of a subharmonic solution depends on history. That is for \( m_2 < m < m_1 \), two attractors coexist; one is the period 1 and the other is the period 2 [4]. Here \( m_2 \) is obtained from the condition for \( b^2 \) to be real (i.e., at \( m = m_2 \), \( b^2 \) has a double root). Explicit expressions for \( m_1 \) and \( m_2 \) are approximately given from (25) as
\[ m_1 = m_2 \sqrt{1 + x^2/3 \alpha^2 \left( 1 + \frac{5}{8} \frac{\sigma}{\omega_0} \right)} \] (26a)
\[ m_2 = \frac{12\alpha \omega_0}{i_b T \left( 1 - \frac{5}{3} \frac{\sigma}{\omega_0} \right)^{1/2}}. \] (26b)
A finite amplitude of the subharmonic that is excited by the parametric resonance shifts the resonance frequency from 1 to its lower frequency side and this shift brings about hysteresis as in the primary resonance of Section II. Therefore, \( m_1 \) exists for \( \sigma < 0 \) (or \( \omega' < 2 \)) and \( m_2 \) becomes smaller as \( \omega' \) is lowered from 2, as is easily seen from (26b). \( m_1 \) and \( m_2 \) are proportional to the damping constant \( \alpha \) that increases with the increases of both \( \beta \) and \( \epsilon \), and inversely proportional to \( T \). Effects of \( T \) and \( \alpha \) on the subharmonic generation is similar to those on the primary resonance. Curves of \( m_1 \) and \( m_2 \) are plotted in Fig. 3, with the driving frequency as a parameter. Predicted values of \( m_1 \) show good agreement with numerical solutions of (1) represented by dots in Fig. 3. They are obtained by solving (1) numerically. However, the error in \( m_1 \) becomes appreciable as \( \omega' \) is lowered away from 2, since it is obtained by assuming \( \sigma = O(\epsilon) \). As the magnitude of \( \sigma \) increases, the nonzero value of \( b \) at \( m_1 \) becomes larger. For \( b > 1 \), (25) becomes inaccurate.
We also performed numerical calculations of (1) and compared them with the analytical results given by (25), as shown in Fig. 4 (a) and (b). As the bifurcation point is approached, convergence of numerical solutions becomes so poor that our numerical calculations yield less accurate values. On the other hand, as \( m \) goes far away from the bifurcation point, values of \( b \) become much greater.
than 1 so that (25) becomes less accurate [(25) is valid for \( b = O(1) \)]. However, analytical results for \( b \) much greater than one still show reasonably good agreement with numerical results. In Fig. 4(b), analytic curves show that, with an increase of \( \alpha \), the minimum value of \( m \) for subharmonic generation increases and the hysteresis region becomes narrower.

Experimental observations of the period 2 solution in pump-modulated lasers have been made [5], [9], [19], [20], while the coexistence of a fundamental and a period 2 solution has not been observed by experiment.

IV. GENERATION OF THE HIGHER SUBHARMONICS

To consider the mechanism for generation of the period 4 solution, we start with the period 2 solution \( u(\xi) = b \cos(\omega' \xi/2 + \theta) \) given in Section III and perturb it by \( \eta \) such that \( \eta \ll u(\xi) \) [16]. Then, we obtain for \( \eta \)

\[
\frac{d^2 \eta}{d \xi^2} + 2\varepsilon \left[ \alpha - b \cos \left( \frac{\omega' \xi}{2} + \theta \right) \right] \frac{d \eta}{d \xi} + \left[ 1 + \frac{\varepsilon b}{2} \sin \left( \frac{\omega' \xi}{2} + \theta \right) \right] \eta = 0
\]

(27)

where the terms that are small \( O(\varepsilon) \) or do not contribute directly to the generation of period 4 solution are neglected for simplicity. It is convenient to transform (27) into the form of Mathieu’s equation by introducing

\[
\eta = \chi \exp \left[ -\varepsilon \alpha \xi + \frac{eb}{2} \int_{0}^{c} \cos \left( \frac{\omega' \zeta}{2} + \theta \right) d \zeta \right]
\]

(28)

which eliminates the damping terms. The transformation yields

\[
\frac{d^2 \chi}{d \xi^2} + \left[ \gamma^2 + \frac{cb}{4} \sin \left( \frac{\omega' \xi}{2} + \theta \right) \right] \chi = 0
\]

(29)

where \( \gamma^2 = 1 - \varepsilon^2 b^2/8 \). Although (29) is valid only for small \( \chi \), it is sufficient to determine the value of \( b \) at which
a nontrivial solution of \( \chi \) is generated. Small perturbations from the period 2 solution will experience a shift of resonance frequency (from \( \gamma = 1 \)) that is equal to \( -\varepsilon^2 b^2 / 8 \), if the amplitude \( b \) of the period 2 solution increases. It is well known that an oscillator, driven parametrically at twice its resonance frequency, exhibits strong resonance (e.g., [21, p. 81]). Thus it is expected from (29) that a parametric resonance will occur when the shifted resonance frequency \( \gamma \) is approximately \( \omega' / 4 \) (i.e., \( (\omega' / 4)^2 = 1 - \varepsilon^2 b^2 / 8 \)).

It is important to note that for the same amplitude the resonance frequency shift of \( \varepsilon^2 b^2 / 8 \) is three times that in the primary resonance (i.e., \( \Delta \omega' = \varepsilon^2 a^2 / 24 \)). In other words, the resonance frequency shift for the small perturbations from a steady-state solution (not a static solution) is larger than the resonance frequency shift due to the large-signal modulation (or the primary resonance). This is important for the period doubling bifurcations, since a larger resonance frequency shift of the small perturbation implies the parametric resonance at a smaller value of modulation index.

Now we solve (29). Although \( \chi = 0 \) is a solution to (29), this solution is unstable for certain parameter ranges. There exist subharmonic solutions to (29) that grow exponentially with time. The solution of (29) may be sought (as [21, p. 82])

\[
\chi = c(\xi) \cos \left( \frac{\omega' \xi}{4} + \frac{\theta}{2} \right) + d(\xi) \sin \left( \frac{\omega' \xi}{4} + \frac{\theta}{2} \right).
\]

(30)

Substituting (30) into (29), we obtain two linear differential equations for \( c(\xi) \) and \( d(\xi) \). Then, we seek solutions proportional to \( e^{\lambda \xi} \). This leads to

\[
\begin{bmatrix}
-\frac{\omega'}{2} + \frac{ebw'}{8} & 1 - \frac{\omega'^2}{16} - \frac{\varepsilon^2 b^2}{8} \\
1 - \frac{\omega'^2}{16} - \frac{\varepsilon^2 b^2}{8} & \frac{\omega'}{2} + \frac{ebw'}{8}
\end{bmatrix}
\begin{bmatrix}
c \\
d
\end{bmatrix}
= \begin{bmatrix}
0 \\
0
\end{bmatrix}.
\]

(31)

For the nontrivial solution of (31) to exist, we obtain

\[
\left( \frac{\omega'}{2} \lambda \right)^2 = \left( \frac{ebw'}{8} \right)^2 - \left( 1 - \frac{\omega'^2}{16} - \frac{\varepsilon^2 b^2}{8} \right)^2.
\]

(32)

The condition for parametric resonance is that \( \lambda \) is real, i.e., \( \lambda^2 > 0 \). This gives the minimum value \( b \) of the first subharmonic amplitude \( b \) for the parametric resonance in (29), or the generation of the period 4 solution in the rate equations (1), as

\[
b^2 = \frac{8 TS_{\alpha}}{1 - \sqrt{1 - \left( 1 - \frac{\omega'^2}{16} \right)^2}}.
\]

(33)

Although the expression of the minimum \( b \) given by (33) has no explicit dependence on both the spontaneous emission factor \( \beta \) and the gain saturation coefficient \( \varepsilon_g \), the modulation current for generation of the period 4 solution depends strongly on \( \beta \) and \( \varepsilon_g \), since the value of \( b \) itself depends strongly on \( \alpha \). It may be noted that (33) also gives the regime of detuning for period 4 generation if we regard \( b \) as a given quantity. The minimum \( b \) given by (33) agrees well with numerical solutions of the rate equations, as shown in Fig. 5. The small error may originate from the fact that in (27) we neglected, for simplicity, nonlinear terms that are of the order of \( \varepsilon \).

The mechanism for generation of the period 8 solution may be explained as follows. The amplitude increases of the period 2 and period 4 solutions bring about the resonance frequency shift of the small perturbation. When the resonance frequency is shifted to about 1/8 of the driving frequency, the oscillator exhibits strong resonance with the period 4 solution which excites the oscillator parametrically. Thus the period 8 solution is generated. Further period doubling bifurcations can be explained in a similar way.

V. DISCUSSION AND CONCLUSION

Since the minimum instantaneous current reaches below or close to the threshold current level and the photon density becomes highly spiky as period doublings occur, chaotic phenomena in laser diodes are expected to depend critically on both the spontaneous emission [4] and the gain saturation [5]. The spontaneous emission factor and the gain saturation coefficient, which act as damping factors in the small-signal analysis of rate equations, suppress instabilities of laser diodes. We have shown in Sections III and IV that generation of the period 2 and period 4 solutions depends strongly on the damping constant. The result given in Section II for primary resonance describes the spiky behavior of the photon density that is utilized in picosecond optical pulse generation [22]. It is also possible to calculate the pulsewidth analytically. We have given essential features of the large-signal modulation characteristics of the laser diode with respect to the electron density. However, it is easy to calculate the corresponding photon density since it is just an integration of the electron density [13].

Now we discuss experimental observations of period doublings and chaos in directly-modulated laser diodes

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Fig. 5. Minimum value \( b_{\text{min}} \) of \( b \) for period 4 oscillation at \( \omega' = 1.9 \) with the parameters the same as in Fig. 1(a). Dots represent numerical results, while the solid line represents the analytical result.
that were reported in [5]. In the absence of RF modulation, lasers exhibit resonance peak in the noise spectrum. When a laser is modulated at a frequency above the intrinsic resonance frequency, the resonance peak shifts towards lower frequencies as the modulation current is increased, as explained in Section II. When the frequency of noise peak is moved to about a half of the drive frequency, the noise peak begins to sharpen. That is, the first subharmonic is about to be generated. However, no further bifurcations have been observed yet in nonlinear-pulsing laser diodes. If the modulation current is increased further, the oscillation gradually becomes period 1. It seems that most lasers have values of damping factor too large to exhibit further period doublings. Thus for the observation of further period doublings it is required to reduce the damping factor, by employing lasers with smaller spontaneous emission factor, or by making lasers weakly-pulsating. Observation of period 4 oscillations in a weakly-pulsating laser was reported [5], where self-pulsation was induced by optical damage. Further period doublings have not been observed clearly yet. It may be attributed to the damping factor of the laser that is still too large to produce period 8 oscillations [4]. Also, fluctuations inhibit observation of higher subharmonics [6].

The observation of a period doubling route to chaos in a Nd-doped fiber laser is also attributed to a very small spontaneous emission factor and gain saturation coefficient [9]. The analytical results given here may be directly applied to other pump-modulated lasers whose dynamic behavior is described by the rate equations.

In conclusion, we applied a standard perturbation theory, the method of multiple scales, to rate equations of semiconductor laser diodes that take into account spontaneous emission and gain saturation, to study resonance frequency shift, bistability, and period doubling bifurcations in directly-modulated laser diodes. Our perturbation analysis yielded explicit expressions of parameter ranges for resonance frequency shift, bistability, and period 2 and period 4 oscillations. We also discussed physical mechanisms of resonance frequency shift and period doubling bifurcations, as well as factors that are critical to their occurrence.

Note Added in Proof: If we define \( m \) such that \( i_n = (i_0 - i_\text{th}) m \cos(\omega t') \) with \( i_\text{th} = 1 + n_0, s_0, \) and \( T \) do not appear explicitly in (3). Therefore, analytic expressions become simpler.

REFERENCES


Chang-Hee Lee, for a photograph and biography, see p. 884 of the May 1989 issue of this journal.

Sang-Yung Shin (M’79), for a photograph and biography, see p. 884 of the May 1989 issue of this journal.