Perturbation Solution of Self-Pulsing in Semiconductor Lasers with a Saturable Absorber

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Abstract—We solve the three-component rate equations for semiconductor lasers with a saturable absorber or for multisection semiconductor lasers by using a singular perturbation method. The effects of nonlinear gain and spontaneous emission are included in the rate equations. By transforming the rate equations to the generalized coordinates, we eliminate the most rapidly varying term adiabatically for the fast saturable absorber. Then, we solve the two-component nonlinear equations to obtain analytic expressions for parametric dependence of self-pulsing amplitude and self-pulsing frequency. The self-pulsing frequency shifts from the small-signal oscillation frequency to the lower-frequency side as we increase the self-pulsing amplitude. The square of the self-pulsing frequency does not linearly depend on the injection current, in agreement with experimental observations. We also derive an optimum saturable absorber recovery time for the shortest optical pulse generation.

I. INTRODUCTION

It is well known that self-pulsing semiconductor lasers are used in compact disk and optical disk pickups. Self-pulsing lasers have been used to generate stable optical pulses of the order of 10 picoseconds [1], while shorter optical pulses may be generated by using the mode locking of self-pulsing lasers [2]. Recently, a tunable microwave frequency modulator in a subcarrier light wave link [3] and an optical clock recovery circuit in a digital optical communication link [4] were realized by using self-pulsing semiconductor lasers.

Self-pulsing is caused by the saturable absorber in the active medium of the semiconductor laser. Long-lived semiconductor lasers and artificially degraded semiconductor lasers show self-pulsing [5]–[7]. The saturable absorption center is produced near the facet during the degradation. Some modifications of laser structure may be made to develop self-pulsing. The unpumped region in the multisection semiconductor laser operates like the saturable absorber [8], [9]. Implantation of protons or ions in the facet of the semiconductor laser may also produce the saturable absorber with a fast recovery time [8].

Dynamics of self-pulsing in semiconductor lasers, may be explained by the rate equations with the saturable absorber [5]–[10]. There exist simple two-component rate equations that model the saturable absorber as an intensity-dependent loss term. However, three-component rate equations that include time-dependent characteristics of saturable absorber are required to study the full dynamics of self-pulsing. While the existence of self-pulsing may be shown by a linear stability analysis, the numerical calculation is required to get the self-pulsing amplitude and the self-pulsing frequency.

Previously, we solved systematically the two-component rate equations for relaxation oscillation [11], large-signal modulation responses [12], and self-pulsing [13] in semiconductor injection lasers by using a singular perturbation method. From our transient solution for the semiconductor laser [11], we showed that the small-signal resonance frequency of the semiconductor laser is shifted to the lower-frequency side as the modulation amplitude of the photon density increases. We also explained the nonlinear large-signal modulation dynamics of the semiconductor lasers, including period doubling bifurcations and bistabilities [12].

In this paper we extend our method of solution to three-component rate equations with a saturable absorber density term, and obtain an analytic solution for self-pulsing. The rate equations include the effects of nonlinear gain and spontaneous emission. In Section II, we describe the rate equations and their static solutions. In Section III, we analyze the stability of the static solution. We derive the condition for self-pulsing and the optimum value of saturable absorber parameters for the shortest optical pulse generation. In Section IV, we transform the rate equations to the generalized coordinates, apply the slaving principle to the fastest coordinate, and derive two-component nonlinear equations. We solve the two-component nonlinear equations by using a singular perturbation method, i.e., a multiple scale expansion. In Section V, we discuss our analytic results and comment about the modulation response.

II. RATE EQUATIONS AND STATIC SOLUTIONS

The dynamics of semiconductor lasers with saturable absorbers can be described by the following rate equations [5]–[10]:

\[ \frac{dN}{dt} = \frac{I}{ed} - \frac{N}{\tau_s} - f(N,S)S \]  
(1a)

\[ \frac{dS}{dt} = f(N,S)S - \frac{S}{\tau_p} - \frac{\sigma N_0 S}{\tau_s} + \frac{\beta N}{\tau_s} \]  
(1b)

\[ \frac{dN_a}{dt} = \frac{N^0 - N_a}{\tau_s} - \sigma N_a S \]  
(1c)

where \( N \) is the electron density in the active region, \( N^0 \) the unsaturated saturable absorber density distributed over the active region of the semiconductor laser, \( N_a \) the saturable absorber density, and \( S \) the photon density. As photons get
absorbed in the cavity, the saturable absorber density $N_a$ decreases. It brings about the decrease of the photon absorption by the saturable absorber. $J$ is the injected current density into the active region, $e$ the electron charge, and $d$ the thickness of the active region. The carrier lifetime, the absorber recovery time, and the photon lifetime are denoted by $\tau_e$, $\tau_a$, and $\tau_p$, respectively. The gain $f(N, S)$ is a function of the electron density and the photon density. The absorption cross section is denoted by $\sigma$, and $\beta$ is the ratio of spontaneous emission coupled into the lasing mode.

Equation (1) describes the single-mode dynamics of a semiconductor laser with saturable absorber. However, the mode frequency spacing of the semiconductor laser is wide compared with the inverse of the coherence time of the single-mode light output. Thus, for multimode lasers the photon density $S$ may also be treated as the total photon density in the active region, unless we are interested in the dynamics of the individual modes. In (1) we assume for simplicity that the confinement factor of the optical field is equal to unity. The gain coefficient $f(N, S)$ is given by $g(N - N_m)(1 - S/S_p^0)$, where $g$ is the differential gain coefficient, $N_m$ is the transparent electron density, and $S_p^0$ represents the strength of the nonlinear gain [14–16].

The dynamics of two-section or multisection semiconductor lasers [8] may also be described by (1) with modified saturable absorber parameters. For these lasers the unsaturated saturable absorber density is given by $N^0 = (1 - h)(N_{m2} - J_s\tau_{s2}/ed)$, where $h$ denotes the ratio of the active region volume to the total laser cavity volume, and $N_{m2}$, $J_s$, and $\tau_{s2}$ are the transparent electron density, the injection current, and the carrier lifetime for the absorber section, respectively. The unsaturated saturable absorber density $N^0$ and the carrier lifetime $\tau_{s2}$ of the absorber section may be adjusted by changing the current that is injected to the absorber section. The variables $N_a$, $\sigma$, and $\tau_a$ in (1) denote the carrier density minus the transparent carrier density, the differential absorption coefficient, and the carrier lifetime of the absorber section, respectively. The injection current $J$ and the transparent carrier density $N_m$ of the gain section are reduced by a factor of $h$.

Normalization of (1) yields

$$\frac{dn}{d\tau} = J - n - (n - N_m)(1 - \epsilon_g)s \quad (2a)$$

$$\frac{ds}{d\tau} = \frac{\tau_e}{\tau_p}(n - N_m)(1 - \epsilon_g)s - s - n_a\sigma + \beta n \quad (2b)$$

$$\frac{dn_a}{d\tau} = \frac{\tau_e}{\tau_a}[n^0 - n_a] - \frac{\sigma}{g}n_a s \quad (2c)$$

where $n = g\tau_p N_a n_a = \sigma\tau_p N_a, s = g\tau_s S, j = g\tau_s\tau_J J/ed$, and $\tau = J/\tau_s$. The normalized nonlinear gain $\epsilon_g$ and the normalized unsaturated saturable absorber density $n^0$ are given by $1/(g\tau_s S_p^0)$ and $\sigma\tau_p N^0$, respectively.

To derive the static solution, we equate time derivatives on the left hand side to zero. For simplicity, we neglect effects of the spontaneous emission and the nonlinear gain. A trivial static solution of (2) is the solution with zero static photon density. If the static photon density is equal to zero, the static electron density $n_0$ is equal to the injection current $j$, and the static saturable absorber density $n_{a0}$ is equal to the unsaturated saturable absorber density $n^0$. This solution is stable when the injection current is below the lasing threshold. Above the lasing threshold, the solution is unstable, and the static photon density has a finite value representing the lasing state. For nonzero static photon density $s_0$, the static electron density and the static saturable absorber density are given by

$$n_0 = 1 + \bar{n}_{th} + n_{a0}, \quad n_{a0} = \frac{n^0}{1 + s_0/s^*} \quad (3)$$

where $\bar{n}_{th} = 1 + n_m$ is the threshold carrier density of the semiconductor laser without the saturable absorber, and $s^* = g\tau_s/\sigma\tau_a$ is the normalized saturation photon density of the saturable absorber. We can increase the saturation photon density by decreasing the absorber recovery time or the absorption cross section.

The static photon density is given by (see (4) below). Since the photon density is real and positive, the static photon density has a single nonzero solution if the saturable absorber parameters satisfy the following relation [8]:

$$s^* = \frac{g\tau_s}{\sigma\tau_a} \geq \frac{n^0}{1 + n^0} = \frac{\sigma\tau_p N^0}{1 + \sigma\tau_p N^0} \quad (5)$$

and only a + sign in (4) gives a physically meaningful solution. The threshold current density for lasing is given by

$$j_{th} = n_{th} = \bar{n}_{th} + n^0 \quad (6)$$

The saturable absorber increases the threshold current. In other words, the increase of absorption cross section $\sigma$ or the amount of unsaturated absorber density $n^0$ increases the threshold current. However, the threshold current is independent of the absorber recovery time. For a semiconductor laser with saturable absorber that satisfies (5), the qualitative feature of the static light output versus the injection current curve ($I$ vs $J$ curve) differs little from that of a semiconductor laser without saturable absorber, as shown in Fig. 1. In other words, (5) is a condition for the saturable absorber term to be a small perturbation.

An interesting feature, optical bistability, can be observed when the condition of (5) is reversed [8]. In this case, the static photon density can have three values, i.e., two nonzero values and zero static photon density, at the same bias current. If we increase the bias current from zero, the semiconductor laser switches from the nonlasing state to the lasing state at a certain bias current $j_{up}$. If we decrease the bias current, then, the semiconductor laser switches from the lasing state to the

$$s_0 = j - \bar{n}_{th} - s^*(1 + n^0) \pm \sqrt{(j - \bar{n}_{th} - s^*(1 + n^0))^2 + 4s^*(j - \bar{n}_{th} - n^0)}$$

(4)
nonlasing state at a different bias current $J_{down}$ that is smaller than $J_{up}$. Thus between the bias current $J_{down}$ and $J_{up}$, the static photon density has two stable values.

III. LINEAR STABILITY ANALYSIS

The static solution is called stable, if small perturbations from the static solution decay to zero as time goes to infinity. Only such a stable static solution can be observed experimentally. To check the stability of the static solution, we perform the linear stability analysis for a given static solution. This method has been used to show the self-pulsing instability in semiconductor lasers with the saturable absorber [1], [5]–[9]. For that purpose, we set

$$n = n_0 + \Delta n, \quad s = s_0 + \Delta s, \quad n_a = n_{a0} + \Delta n_a$$  \(7\)

where $\Delta n, \Delta s,$ and $\Delta n_a$ are perturbations from the given static solution $n_0, s_0,$ and $n_{a0},$ respectively. Substitution of (7) into (2) yields the following equations for the perturbations:

$$\frac{d}{d\tau} \begin{pmatrix} \Delta n \\ \Delta s \\ \Delta n_a \end{pmatrix} = \begin{pmatrix} -1 & -1 & 0 \\ -T_1 s_0 & -T_1 s_0 & -T_1 s_0 \\ -T_2 + \sigma & -T_2 + \sigma & -T_2 + \sigma \end{pmatrix} \begin{pmatrix} \Delta n \\ \Delta s \\ \Delta n_a \end{pmatrix} + \begin{pmatrix} -\epsilon_g (n_{th} - n_m) \Delta s + j_1 \\ T_3 (\Delta n - \epsilon_g (n_{th} - n_m) \Delta s) \Delta s \\ -\frac{\sigma}{g} \Delta n_a \Delta s \end{pmatrix}$$  \(8\)

where $T_1 = \tau_s/\tau_p, T_2 = \tau_s/\tau_a, T_3 = T_1 [\beta n_{th} / s_0 + \epsilon_g n_{th} (n_{th} - n_m)],$ and $j_1$ is the modulation current. Here, the effects of spontaneous emission and nonlinear gain are considered as small perturbations, since they are important to understand stability of semiconductor lasers [1], [11]. The parameter $T_3$ incorporates such effects. For the linear stability analysis neglecting the nonlinear perturbation terms, we derive the following characteristic equation by imposing the condition of nontrivial solution in (8).

$$\lambda^3 + \lambda^2 \left[ 1 + s_0 + T_2 + \frac{\sigma}{g} s_0 + T_3 \right] + \lambda \left[ (1 + s_0) \left( T_2 + \frac{\sigma}{g} s_0 \right) + (1 + s_0 + T_2 + \frac{\sigma}{g} s_0) T_3 \right] + T_1 s_0 \left[ 1 + n_{a0} \left( 1 - \frac{\sigma}{g} \right) \right] + T_1 s_0 \left[ T_2 + \frac{\sigma}{g} s_0 + n_{a0} \left( T_2 - \frac{\sigma}{g} \right) \right] + T_3 (1 + s_0) \left( T_2 + \frac{\sigma}{g} s_0 \right) = 0$$  \(9\)

Since (9) is a third-order algebraic equation, it has three eigenvalues. For a saturable absorber that satisfies (5), the characteristic equation has one real eigenvalue $\lambda_3$ and a pair of complex eigenvalues $\lambda_{1,2} = -\alpha \pm i \omega_0$ for the typical parameters of semiconductor lasers ($T_1 s_0 \sim 1, 000$). It may be noted that in the limit of $T_2 \rightarrow \infty$ the characteristic equation becomes that of the semiconductor laser without the saturable absorber. It also has a pair of complex eigenvalues [11], and they are related to the eigenvalue of the semiconductor laser rate equations without the saturable absorber. In other words, the real part of the complex eigenvalue is related to the damping constant of the semiconductor laser rate equations, and the imaginary part is related to the small-signal resonance frequency [1], [11], [13]. The nonlinear gain and the spontaneous emission increase the damping constant (i.e., they decrease the resonance peak in the modulation response), while the saturable absorber decreases the damping constant and eventually develops self-pulsing [13].

The real eigenvalue is always negative, since the coefficients of the $\lambda^3$ term and the $\lambda^2$ term are positive. Thus the unstable static solution occurs when the real part of the complex eigenvalue becomes positive. This implies a possibility of self-pulsing or the Hopf bifurcation. The Routh–Hurwitz criteria for stability show that the characteristic equation has a pair of unstable complex eigenvalues under the following condition:

$$\begin{pmatrix} 1 + s_0 + T_2 + \frac{\sigma}{g} s_0 \\ (1 + s_0) \left( T_2 + \frac{\sigma}{g} s_0 \right) \end{pmatrix} + T_3 \left[ 1 + s_0 + T_2 + \frac{\sigma}{g} s_0 + T_3 \right] + T_1 s_0 \left[ (1 + s_0)(1 + n_{a0}) + T_3 \left( 1 + n_{a0} \left( 1 - \frac{\sigma}{g} \right) \right) \right] - T_1 T_2 \frac{\sigma}{g} s_0 = 0$$  \(10\)

A large absorption cross section $\sigma$ or a large amount of unsaturated absorber density $n^0$ increases the possibility of self-pulsing. However, the increase of those parameter values also enlarges the threshold current.

Near the self-pulsing threshold, the real part of the complex eigenvalue is approximately zero. Thus we can solve the characteristic equation of (9). The small-signal self-pulsing
frequency (i.e., the imaginary part of the complex eigenvalues) is given by
\[
\omega_0^2 = T_1 s_0 \left[ 1 + na_0 \left( \frac{1 - \sigma}{g} \right) + (1 + s_0) \left( T_2 + \frac{\sigma}{g} s_0 \right) \right] + T_3 \left( 1 + s_0 + T_2 + \frac{\sigma}{g} s_0 \right). \tag{11}
\]

The small-signal self-pulsing frequency increases as the saturable absorber recovery time decreases. It may be noted that \( T_1 s_0 \) is the square of the small-signal resonance frequency of a semiconductor laser without saturable absorber. The real eigenvalue \( \lambda_3 \) is approximately the coefficient of the \( \lambda^2 \) term with negative sign. Near the self-pulsing threshold, the real part of complex eigenvalues may be analytically derived by the iteration method. If the static photon density is far from the self-pulsing threshold, it is required to solve the characteristic equation of (9) numerically.

Fig. 1 shows the calculated \( L-I \) curve and eigenvalues as a function of the injection current. The self-pulsing starts at a certain value of bias current (where the real part of complex eigenvalue becomes positive), and it stops at a larger static photon density. It may be noted that the introduction of saturable absorber as an intensity dependent loss term, i.e., \( n^0(1 - s/s^*) \), in the two-component rate equations does not predict the cease of self-pulsing at large static photon density [13]. Fig. 2 shows the magnitude of the real eigenvalue \( \lambda_3 \) and the imaginary part of the complex eigenvalue, i.e., the small-signal resonance frequency \( \omega_0 \). The real eigenvalue is larger than the small-signal resonance frequency for the fast saturable absorber.

In Fig. 3 we show the maximum value of the static photon density for self-pulsing versus the inverse of the absorber recovery time, i.e., \( T_2 \). As we increase \( T_2 \), the maximum value of the static photon density for self-pulsing increases, reaches at a peak, and decreases eventually. Since the small-signal self-pulsing frequency is approximately proportional to the square root of the static photon density [see (11)], self-pulsing at the highest value of the static photon density implies a possibility of self-pulsing with the shortest optical pulse width and with the highest peak power. The value of \( T_2 \) at the peak, \( \bar{T}_2 \), and the corresponding maximum static photon density \( \bar{s}_0 \) are given by
\[
\bar{T}_2 \simeq \frac{T_1 \bar{s}_0 n^0 \sigma/g}{2(1 + \bar{s}_0 + T_3)} - \frac{1 + \bar{s}_0 + T_3 + 2\bar{s}_0 \sigma/g}{2}, \tag{12a}
\]
\[
\bar{s}_0 \simeq \frac{T_1 (\sigma/g)^2 (n^0)^2}{4[1 + T_1 \sigma/(n_{th} - n_m)][1 + (\sigma/g)^2 n^0]}, \tag{12b}
\]

These values increase with the absorption cross section or the amount of the unsaturated absorber density. The nonlinear gain and the spontaneous emission decrease both \( \bar{T}_2 \) and \( \bar{s}_0 \). For a semiconductor laser with \( n^0 = 0.1 (N^0 = 2 \cdot 10^{-16} \text{cm}^{-3}) \), \( \sigma/g = 5 \), \( T_1 = 1000 \) (\( \tau_e = 1 \text{ ns} \)), and \( \tau_p = 1 \text{ ps} \), \( \beta = 10^{-5} \), and \( \epsilon_g = 10^{-3} \) (nonlinear gain \( S_g^* = 10^{-18} \text{cm}^{-3} \)), the optimum value of absorber recovery time is approximately \( 10^{-2} \) times the carrier lifetime, e.g., 10 ps for gain medium with \( \tau_e = 1 \text{ ns} \). The maximum value of photon density is 1.78 (i.e., \( S_0 = 1.78 \cdot 10^{15} \text{cm}^{-3} \)). In the limit of zero saturable absorber lifetime (i.e., \( T_2 \to \infty \)), the saturation photon density of the saturable absorber becomes infinite, and the saturable behavior of the absorber is stopped. Thus the self-pulsing is no longer possible.

IV. TRANSFORMATION AND SELF-PULSING SOLUTION

The transformation of nonlinear equations to their generalized coordinates is a commonly used method to solve nonlinear differential equations [11], [12], [17]. The transformation yields a set of nonlinear equations that have only eigenvalues of their small-signal analysis as their coefficients of linear terms. The transformation matrix is formed by the eigenvectors of the linearized equations. We can apply the slaving principle (i.e., the adiabatic elimination of rapidly varying coordinates) and reduce the dimension of nonlinear equations, if there exists a stable eigenvalue whose magnitude is much larger than the magnitudes of the other eigenvalues.

\[
B_2 = \frac{\alpha - 1 - s_0 + i\omega_0}{1 + na_0} \tag{14a}
\]
\[
B_3 = \frac{T_1 s_0[1 + na_0] - \omega_0^2 + (1 + s_0 - \alpha)(T_3 - \alpha) + i\omega_0(2\alpha - 1 - s_0 - T_3)}{T_1 s_0(1 + na_0)} \tag{14b}
\]
\[
b_{23} = -\frac{1 + s_0 + \lambda_3}{1 + na_0} \tag{14c}
\]
\[
b_{33} = \frac{\lambda_3^2 + \lambda_3(1 + s_0 + T_3) + T_3(1 + s_0) + T_1 s_0(1 + na_0)}{T_1 s_0(1 + na_0)} \tag{14d}
\]
We transform the normalized rate equation, (8), to the following generalized coordinates [11]:

$$
\begin{pmatrix}
\Delta n \\
\Delta s \\
\Delta n_0
\end{pmatrix} = B
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix} =
\begin{pmatrix}
1 & 0 & 1 \\
\Re(B_2) & \Im(B_2) & b_{23} \\
\Re(B_3) & \Im(B_3) & b_{33}
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix}
$$

(13)

where $\Re(B_i)$ and $\Im(B_i)$ represent the real part and the imaginary part of argument $B_i$, respectively. The matrix elements of the transformation matrix $B$ are given by (see (14a)-(14d) on previous page). Here, we use a complex eigenvalue of $-\alpha - i\omega_0$, and a real eigenvalue of $\lambda_3$ of the characteristic equation to derive the transformation matrix $B$. After scaling the variables in the generalized coordinate such as

$$
u = T_1 x, \quad \xi = \omega_0 \tau \quad (15)$$

we have the equations of motion for $u$, $v$, and $w$.

$$
\frac{d}{d \xi}
\begin{pmatrix}
u \\
v \\
w
\end{pmatrix} =
\begin{pmatrix}
-\frac{\alpha}{\omega_0} & -1 & 0 \\
1 & -\frac{\alpha}{\omega_0} & 0 \\
0 & 0 & \lambda_3
\end{pmatrix}
\begin{pmatrix}
u \\
v \\
w
\end{pmatrix}
+ B^{-1}
\frac{f_1}{T_1 \omega_0}
\begin{pmatrix}
T_1(f_1 - f_2)
\end{pmatrix}.
$$

(16)

In (16), $B^{-1}$ denotes the inverse matrix of the transformation matrix $B$, and the nonlinear terms are given by

$$f_1 = \Re(B_2)u^2 + \Im(B_2)uv + [b_{23} + \Re(B_2)]uw + \Im(B_2)uv + b_{23}w^2$$

$$- \varepsilon_g (1 + n_{a0})[\Im(B_2)^4 v^2 + 2 \Im(B_2)b_{23}uv + b_{23}^3 w^2]$$

(17a)

$$f_2 = \Re(B_2)\Re(B_3)u^2 + [\Re(B_2)\Im(B_3)$$

$$+ \Im(B_2)\Re(B_3)]uv + \Im(B_2)\Im(B_3)v^2$$

$$+ [b_{23} \Re(B_2) + b_{23} \Re(B_3)]uw + [\Im(B_3)b_{23}$$

$$+ \Im(B_3)b_{33}]uv + b_{23}b_{33}w^2.$$ (17b)

Equation (16) is the transformed equation of (8) to generalized coordinates. For nonlinear terms including the nonlinear gain $\varepsilon_g$, we take only the dominant nonlinear terms.

Typically, $T_1 \approx 1000, s_0 \approx 1(S_0 \approx 10^{10} \text{cm}^{-3})$, and $\sigma/g$ is about 5 for bulk semiconductor lasers and of the order of 10 for quantum-well semiconductor lasers. The parameter $T_2$ ranges from about 1 to 1,000, depending on the lifetime of the saturable absorber. Thus $\omega_0$ is of the order of 10, $\alpha$ is of the order of unity, and $-\lambda_3 \approx T_2 + s_0 \sigma/g$. Now we introduce a small parameter $\varepsilon = 1/\omega_0$ and estimate the order of magnitude of each coefficient. And we neglect the terms that are lower in order than the other terms. For the saturable absorber with a fast carrier recovery time or that with a high value of $\sigma/g$, the magnitude of the real eigenvalue $\lambda_3$ is larger than the magnitude of the complex eigenvalue as shown in Fig. 2. Thus we can eliminate the most rapidly varying variable $w$ adiabatically:

$$w = c_1uv + c_2(\sigma s_0/g \omega_0)u^2,$$ (18a)

$$c_1 = \frac{\omega_0}{\lambda_3(\lambda_3^2 + \omega_0^2)}(2\alpha - 1 - s_0 - T_3).$$ (18b)

In deriving (18), it was noted that $\Re(B_3)$ is nearly zero, since $T_1 s_0 \approx \omega_0^2$ and $n_{a0} \ll 1$. Also we neglect the nonlinear terms involving $w$, since their coefficients are of the order of $\varepsilon^2 = 1/\omega_0^2 \ll 1$ and $w$ is also of the order of $\varepsilon^2$ at most. Then (16) may be reduced to two-component nonlinear equations. Since we know that the self-pulsing amplitude of semiconductor laser is determined by the third-order singular perturbation method [13], we keep nonlinear terms up to the second order.
of the small parameter $\epsilon$ for the second order nonlinear terms in deriving the two-component nonlinear equations. For the cubic nonlinear terms, we keep up to the third-order terms (i.e., $\epsilon^3$ order terms) since they give secular terms in the third-order perturbation expansions. After straightforward algebraic manipulations, we have the following adiabatically eliminated two-component nonlinear equations:

\[
\begin{align*}
\frac{du}{d\xi} &= -\epsilon \alpha u + \epsilon^2 c_2 v u \\
\frac{dv}{d\xi} &= u - \epsilon \alpha v + \epsilon v u + \epsilon^2 (\alpha - s_0) u^2 - \epsilon^2 c_3 u^3 + \epsilon^3 c_4 v^3 + \epsilon^3 c_5 u^2 v
\end{align*}
\]  

where

\[
\begin{align*}
c_2 &= -s_0(1 - T_1(2\alpha - 1 - s_0 - T_3)/(\lambda_3^2 + \omega_0^2)) \\
c_3 &= (2\alpha - 1 - s_0 - T_3)[1 + \sigma s_0 \lambda_3^2/(\lambda_3^2 + \omega_0^2)] + \epsilon \omega_0^2 \\
c_4 &= -(2\alpha - 1 - s_0 - T_3)(\sigma s_0 g_3 \lambda_3^2)[1 - \omega_0^2/(\omega_0^2 + \lambda_3^2)] \\
c_5 &= -(2\alpha - 1 - s_0 - T_3)\omega_0^2/(\omega_0^2 + \lambda_3^2).
\end{align*}
\]

It may be noted that $2\alpha - 1 - s_0 - T_3$ is equal to $-T_1 \sigma s_0^2/\beta$ in the limit of zero absorber lifetime. The variable $v$ in (19) represents the ac component of the electron density, since the variable $w(\epsilon^2)$ or less) is much less than $u(O(1))$. From (19) we have the following nonlinear equation for $u$:

\[
\begin{align*}
\frac{d^2 u}{d\xi^2} + 2\epsilon \alpha \frac{du}{d\xi} + u &= \epsilon u + \epsilon^2 (c_2 - c_3) \left( \frac{du}{d\xi} \right)^2 \\
&+ \epsilon^2 (1 + s_0 + c_2) u^2 \\
&+ \epsilon^3 c_4 \left( \frac{du}{d\xi} \right)^3 + \epsilon^3 c_5 u^2 \frac{du}{d\xi}
\end{align*}
\]  

(21)

It may be noted that (21) is equivalent to (2) in [13], if we neglect the third-order nonlinear terms. The third-order nonlinear term of $(du/d\xi)^2$ arises from the higher-order saturation of the absorber. In other words, the fast saturable absorber can be represented as $n_0^0/(1 + s^*/s^*)$, where $s^* = \sigma \tau_0/\sigma \tau_0 \approx \text{abs} \lambda_3^2 g_3/\alpha$ where abs represents absolute value of the argument. Thus the first-order approximation of the saturable absorber may be $n_0^0(1 - s^*/s^*)$. We used this approximation in [13] to solve rate equations. However, approximation up to higher-order saturation terms, such as $n_0^0(1 - s^*/s^* + (s^*/s^*)^2)$, is required for the self-pulsing solution with a large amplitude. The third nonlinear term $u^2 du/d\xi$ is due to a finite response time of the saturable absorber.

We can solve (21) by using a singular perturbation method, i.e., the multiple scale expansion method, since all nonlinear terms have the small parameter. We already used this method to solve the two-component rate equations for semiconductor lasers. The details are described in [11] and [18]. In summary we first expand $u$ and $\xi$ with small parameter $\epsilon$, and derive the equations for each order of small parameter. Then, we solve the equations successively so that secular terms giving the divergent solution vanish [11, 18].

The multiple scale expansion method yields the following equations for the self-pulsing amplitude $a$ and its phase $\gamma(u = 0.5 a \exp (i \gamma) + c.c.)$, provided that secular terms are equal to zero:

\[
\frac{da}{d\xi} = -\epsilon \alpha a + \epsilon^2 a^2 \left[ \frac{8}{9} \frac{\alpha - T_1 \beta n_{th}}{s_0} - \frac{(2\alpha - 1 - s_0 - T_3)}{\omega_0^2 + \lambda_3^2} \right]
\]  

\[
\frac{d\gamma}{d\xi} = 1 - \epsilon^2 a^2
\]

(22a)

The last term in the brace of (22a) is due to a finite response time of the saturable absorber. The other terms in the brace arise from higher-order saturation of the saturable absorber, as explained before.

To steady-state self-pulsing amplitude $a$ and the self-pulsing frequency $\omega_{sp}$ in $\xi$ coordinate are derived from (22) and given by (see (23a) and (23b) below). Then, the solution of (21) is given by

\[
u = \frac{\epsilon^2}{2}(1 + s_0 - \alpha + c_3) a^2 + a \cos (\omega_{sp} \xi)
\]

\[
+ \frac{\epsilon a^2}{6} \sin (2\omega_{sp} \xi)
\]

\[
- \frac{\epsilon^2 a^3}{32} \cos (3\omega_{sp} \xi) - \frac{13 \epsilon a^4}{2160} \sin (4\omega_{sp} \xi)
\]

(24)

The self-pulsing frequency shifts to the lower frequency proportionally to the square of the amplitude $a$, as the self-pulsing amplitude $a$ increases. This is the same as the resonance frequency shift in the laser rate equations without saturable absorber [11, 12]. Physically, this feature arises from the fact that a large depletion of carrier density due to a large self-pulsing amplitude requires a long recovery time.

We show both the self-pulsing amplitude and the square self-pulsing frequency in $\tau$ coordinate versus the injection current in Fig. 4. For comparison, we also show numerical results which are obtained from the original rate equation, i.e., (2). The analytic results show good agreement with numerical
results. Some errors for large self-pulsing amplitudes arise from slow convergence of the perturbation series.

As we increase the bias current $j$ (or the static photon density $s_0$), self-pulsing starts at a certain bias current $j_0$ (or $s_0$) when the damping constant becomes negative. Eventually self-pulsing stops at the high photon density limit. The maximum value of the static photon density for self-pulsing is shown in Fig. 3. The nonlinear gain in semiconductor lasers has an effect of decreasing both the self-pulsing amplitude and the shift of oscillation frequency. It also requires higher $s_0$ for the onset of self-pulsing. As shown in Fig. 4(b), the square of the self-pulsing frequency does not depend linearly on the injection current, due to shift of the oscillation frequency. The self-pulsing frequency is nearly a constant when the self-pulsing starts. The amplitude and the frequency of self-pulsing increase with the bias current. And the square of the self-pulsing frequency depends linearly on the injection current in the region of well-developed self-pulsing as shown in curves for $\phi = 10^{-4}$ in Fig. 4(b). However, the slope is less than that predicted in the small-signal analysis. These features were observed experimentally [19], [3].

The effects of the spontaneous emission are similar to that of the nonlinear gain. The spontaneous emission also decreases the self-pulsing amplitude and the shift of self-pulsing frequency, since it decreases the magnitude of the damping constant $\alpha$. It also shrinks the self-pulsing region.

In the well-developed self-pulsing region, the minimum obtainable pulse width of the photon density is about 0.16 times the period of small-signal oscillation. Thus self-pulsing at the highest value of the static photon density implies a possibility of self-pulsing with the shortest optical pulse width and with the highest peak power. The optimum value of $T_2$ for self-pulsing at the highest value of small-signal oscillation frequency is given by (12).

Finally, the steady-state solutions for the photon density, the electron density, and the saturable absorber density are obtained from (13), (18), and (24).

\begin{equation}
\eta = n_0 + \frac{\tau_e}{\tau_s} \frac{\tau_s}{\eta_0} \frac{d\eta}{d\xi}
\end{equation}

\begin{equation}
\eta_a = n_0 - \frac{\tau_e}{\tau_s} \text{Im}(B_2) \frac{d\eta}{d\xi}
\end{equation}

\begin{equation}
s = s_0 - \frac{\tau_e}{\tau_s} \text{Im}(B_2) \frac{d\eta}{d\xi}
\end{equation}

We show numerically calculated waveform of the dynamic variables in Fig. 5(a) and the analytically calculated results in Fig. 5(b) for comparison. The analytical results show good agreement with the numerical results.

V. DISCUSSION AND CONCLUSION

First we consider the adiabatic elimination. As shown in (22a) or (23a), the self-pulsing amplitude is a function of the real eigenvalue $\lambda_1$ (or the saturable absorber lifetime) due to a finite response time of the absorber. The self-pulsing amplitude and the shift of oscillation frequency increase as the absorber lifetime increases. It also brings about a large error of the analytic results. We found that the saturable absorber with lifetime of the order of 10 ps is enough for the adiabatic elimination. The saturable absorber made by ion or proton implantation in the semiconductor has about 10-ps recovery time [1]. Also the saturable absorber induced by the degradation of active region may have a similar recovery time for (13)]. The absorber recovery time can be reduced greatly by applying the reverse bias to the absorber region in the multisection semiconductor lasers. For semiconductor lasers with these kinds of saturable absorbers,
the self-pulsing amplitude and the self-pulsing frequency may be expressed by our analytic results. However, it is not easy to compare our theoretical calculation with experimental results. For the critical comparison, a full set of parameters should be known, especially the absorber recovery time and the ratio of the absorption cross section to the differential gain coefficient.

The self-pulsing solution obtained here shows the supercritical Hopf bifurcation (i.e., the self-pulsing amplitude increases continuously from zero as we increase the injection current). Thus, it is hard to expect the subcritical Hopf bifurcation (i.e., bistability in the self-pulsing solution), since the bistability usually accompanies the abrupt transition. More complex dynamics may exist in the semiconductor lasers with a relatively slow saturable absorber section.

The increase of the absorption recovery time or the decrease of the absorption cross section causes the magnitude of the real eigenvalue $\lambda_3$ to decrease. Then, at some point the adiabatic elimination is no longer possible. In this case, we must solve three-component rate equations. Unfortunately, we have not derived analytic solutions for this case. For the semiconductor laser with slow saturable absorber, such as the unpumped region in two-section bulk semiconductor lasers without ion implantation or reverse bias, numerical analysis of three-component rate equations is required to derive self-pulsing amplitude and self-pulsing frequency. From numerical calculations, we found that the self-pulsing amplitude and the self-pulsing frequency shift increase as we increase the absorber recovery time.

Beside the self-pulsing solution, the modulation response is also important for the application of the semiconductor lasers with the saturable absorber. From (8) the small signal modulation response is obtained as

$$\Delta s(i\omega) = \frac{[i\omega + T_3 + s_0\sigma/g]}{i\omega - \lambda_3} T_{1,s_0} \frac{T_1}{(i\omega + \alpha - i\omega_0)(i\omega + \alpha + i\omega_0)} j_1(i\omega)$$  (26)

The expression in the square bracket represents the effect of saturable absorber on the small-signal modulation response of the semiconductor lasers. For the fast saturable absorber or the saturable absorber with an absorption cross section that is large compared with the differential gain coefficient, the square bracket becomes unity. Thus the small-signal modulation response is equivalent to that of the semiconductor laser without the saturable absorber. An important feature is the reduction of the damping constant due to the saturable absorber. It implies that we can overcome the nonlinear gain effects (e.g., the decrease of the modulation bandwidth due to the increase of the damping factor) on the modulation response by introducing a small amount of saturable absorber. It may be noted that the relatively slow saturable absorber with a large absorption cross section can be used to increase the modulation bandwidth, provided that the saturable absorber parameters satisfy (5).

The large-signal modulation response of the semiconductor laser with the saturable absorber can be obtained by solving the semiconductor laser rate equations. The damping factor, i.e., the real part of the complex eigenvalue, is the most important parameter to understand the modulation dynamics. When the damping factor is positive (or the static solution is stable), the results in (12) describe the modulation dynamics of the semiconductor laser with the saturable absorber, since (19) is equivalent to the transformed rate equations of the conventional semiconductor lasers with different damping constant and small signal resonance frequency. It may be noted that the saturable absorber decreases the damping constant, and the semiconductor laser with the saturable absorber is more susceptible to nonlinear resonance and chaos. If the damping constant is negative (i.e., the case of self-pulsing lasers), the modulation dynamics may show frequency locking. This behavior may be explained by applying a singular perturbation method to (19) [18].

In conclusion, we solved the three-component rate equations for semiconductor lasers with the fast saturable absorbers (i.e., saturable absorbers with a recovery time of the order of 10 ps or less) by using a singular perturbation method. The effects of nonlinear gain and spontaneous emission were considered in the rate equations. Explicit algebraic expressions for the parametric dependence on the self-pulsing amplitude and the self-pulsing frequency were derived. The self-pulsing frequency shifts from the small-signal oscillation frequency to the lower-frequency side, as the self-pulsing amplitude increases. This feature explains that the square of the self-pulsing frequency depends nonlinearly on the injection current.

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